N. M. J. Woodhouse

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Introduction to Analytical Dynamics

New Edition

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Introduction to Analytical Dynamics

New Edition

Springer

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Preface to the New Edition

This is a revised edition of a text on classical mechanics that was originally published 20 years ago by Oxford University Press. I have taken the opportunity to simplify some of the presentation, while keeping to the original intention of confronting rather than evading the various notational and pedagogical difficulties that one encounters in the journey from Newton to Lagrange and Hamilton. I have also responded to comments over the years from colleagues and, more recently, from the new publisher's referees.

There are two major changes. I have gathered together the material and examples on systems with one degree of freedom into a separate chapter. The intention here is to give a first introduction to the core ideas of the Lagrangian theory in a context in which they make strong contact with familiar elementary techniques from the treatment of ordinary differential equations, without the distraction of indices and the summation convention. Second, I have added a chapter on differential geometric methods.

I am grateful to the many students and colleagues who have commented on the first edition, and pointed out mistakes.

N.M.J. Woodhouse
Oxford, February 2009

Preface to the First Edition

It may seem odd that Newtonian mechanics should still hold a central place in the university mathematics curriculum. But there are good reasons.

– It is one of the most accurate physical theories ever devised. Three hundred years after the publication of Newton's *Philisophiae naturalis principia mathematica* (1687), we should be surprised not that some of his ideas have been superseded by relativity and quantum theory, but that it is still necessary to exercise great subtlety and scientific ingenuity to detect any error at all in the three laws of motion. Even in the prediction of the orbit of the planet Mercury, for example, which was a crucial failure of the classical theory, the discrepancy[1] is only one part in 10^7.

  Newton's theory is the prime example of what Wigner calls the 'unreasonable effectiveness of mathematics' as a tool for understanding the physical world – an aspect of the truth of mathematics that can easily be lost in a course overburdened with abstraction [14].

– Quantum theory and relativity have overthrown the classical view of physics, but the mathematical formalism of classical mechanics still plays an essential part. It provides both a framework for interpretation and a first introduction to key ideas and techniques (frames of reference, general coordinate transformations, the connection between observables and symmetries,...). It is an essential prerequisite for any advanced course on applications of mathematics in modern theoretical physics.

– It develops geometric intuition and gives invaluable practice in problem

[1] The radius vector from the Sun to Mercury sweeps out a total angle of $150,000^\circ$ per century. The prediction of the Newtonian theory is $43''$ less than the observed angle. For the other planets, it is much less.

solving and mathematical modelling. It is easy to poke fun at the seemingly endless supply of light rods, inextensible strings, and smooth hemispheres. But all undergraduate exercises are necessarily artificial, however cleverly they are dressed. The strength of mechanics is the vast range of its examples, something that their familiarity can make us overlook, and the diversity of different mathematical ideas that they illustrate.

– The problems of classical mechanics and, in particular, the centuries of work on planetary motion, stimulated the development of much of modern mathematics. It is no coincidence that the great names of mechanics, Newton, Euler, Poisson, Lagrange, Hamilton,..., also occur over and over again throughout many branches of pure mathematics. It is essential to study classical mechanics to understand the roots of mathematics.

– The influence of classical mechanics is still present in modern pure mathematical research. The study of Hamilton's equations, for example, led to the development of symplectic geometry, which in turn has found recent applications in the analysis of partial differential equations and in the representation theory of Lie groups.

A glance through the pages that follow will not reveal anything strikingly unfamiliar. The range of topics is central and traditional, partly because I want the book to be short and (OUP willing) cheap, and partly because I intend it to be no more than an introduction. I hope that it will be read in conjunction with the classics and that it will encourage further exploration (in, for example, Arnol'd's *Mathematical methods of classical mechanics* [1]).

The book is written for second year mathematics undergraduates and assumes familiarity with elementary linear algebra, the chain rule for partial derivatives, and vector mechanics in three dimensions (the last is not absolutely essential). The main intention is, first, to give a confident understanding of the chain of argument that leads from Newton's laws through Lagrange's equations and Hamilton's principle to Hamilton's equations and canonical transformations; and, second, to give practice in problem solving. Most of the exercises and examples are taken from recent Oxford examination papers.

I have concentrated on trying to clarify the points that come up most frequently in tutorials and that I myself found confusing when I first met these ideas. For example: why are you allowed to say that q and $\dot{q}$ are independent? and: why can I not deduce from $\partial L/\partial t = -\partial h/\partial t$ that $h + L$ is independent of t?

It is true, of course, that the most satisfactory way to come to terms with the mathematics of classical mechanics would be to approach the subject from modern differential geometry. But that would mean reducing analytical mechanics to a minority option at the end of the undergraduate course or in the first year of graduate work, which would be a great loss. Instead, I have tried to make use of lessons that I have learnt from differential geometry, but without ever going outside the framework of local, coordinate-based arguments.

I am particularly grateful to Paul Tod and Tom Cooper for many comments on an earlier version of this book; and to Rob Baston, Andy Clark, Mike Dobson, Steve Lloyd, Diana Mountain, Charles Sanderson, and Steve Thorsett for working through the final version.

Oxford 1986
N.M.J. Woodhouse

Note. Examples and exercises marked with a dagger (†) are adapted from examination questions set at the University of Oxford.

Contents

1
Frames of Reference

1.1 Introduction

The solution to a mechanical problem begins with the kinematic analysis, the analysis of how a system can move, as opposed to how it actually does move under the influence of a particular set of forces. In this first stage, the essential step is the introduction of coordinates to label the configurations of the system. These might be Cartesian coordinates for the position of a particle, or angular coordinates for the orientation of a rigid body, or some complicated combination of distances and angles. The only conditions are that each physically possible configuration should correspond to a particular set of values of the coordinates; and that, conversely, the coordinates should be *independent*, which can be understood informally to mean that each set of values of the coordinates should determine a unique configuration. The number of coordinates is called the *number of degrees of freedom* of the system.

Example 1.1

A particle moving in space has three degrees of freedom. Its position is determined by three Cartesian coordinates or by three spherical polar coordinates.

Example 1.2

A particle moving on the surface of a sphere has two degrees of freedom, labelled by the two polar angles θ and φ (Figure 1.1)

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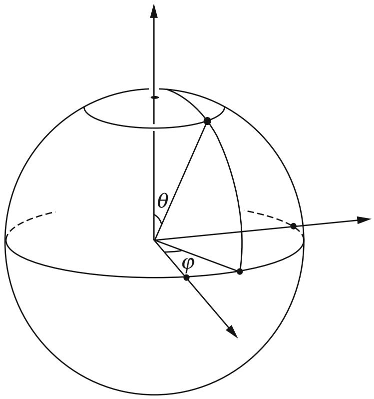

Figure 1.1

Example 1.3

Two particles connected by a rigid rod have five degrees of freedom: if the position of one particle is given (three coordinates), then the other can be anywhere on a sphere with its centre at the position of the first particle (two further coordinates).

Example 1.4

A rigid body has six degrees of freedom: three for the position of the centre of mass; two for the direction of some axis fixed in the body; and one for rotations about this axis.

The second stage, the dynamical part of the problem, is to use Newton's second law to determine the actual motion: to find out how the coordinates evolve as functions of time when the system is subjected to given forces. In the next few chapters we look at a number of techniques for finding and solving dynamical equations in general coordinate systems. These make it possible to simplify the second stage of a variety of mechanical problems, particularly problems involving constraints, by choosing well adapted coordinates in the first stage.

First, however, let us get our bearings by considering a very simple problem, the motion of a single particle moving in space under the influence of a given force. The kinematic analysis is easy. We can describe the motion of the particle by introducing a frame of reference R, which defines a standard of rest, but we need to think about what freedom is available in the choice of frame, what other coordinate systems could be used, and about how they are related.

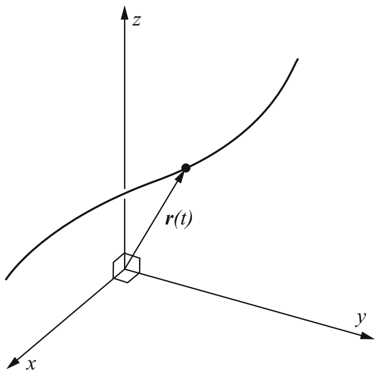

Figure 1.2

Definition 1.5

A *frame of reference* is an origin together with a set of right-handed Cartesian coordinate axes.

Let $\boldsymbol{r}$ denote the particle's position vector from the origin of R. Then the components of $\boldsymbol{r}$ along the axes, x, y, and z, are its Cartesian coordinates. A motion of the particle is represented by a curve $\boldsymbol{r} = \boldsymbol{r}(t)$ in space, along which x, y, and z are functions of time (Figure 1.2).

Definition 1.6

The *velocity* and *acceleration* of the particle relative to R are the vectors $\boldsymbol{v}$ and $\boldsymbol{a}$ with respective components $(\dot{x}, \dot{y}, \dot{z})$ and $(\ddot{x}, \ddot{y}, \ddot{z})$, where the dot denotes the derivative with respect to time.

It is important to remember that 'velocity' and 'acceleration' do not make sense unless one adds, either explicitly or by implication, 'with respect to such-and-such a frame of reference'.

Turning to the dynamics, it is an axiom of Newtonian mechanics that there exist special frames of reference, called *inertial frames*, in which Newton's second law holds. If R is such a frame, then

$$m\boldsymbol{a} = \boldsymbol{F} \tag{1.1}$$

where m is the mass of the particle and $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v}, t)$ is the force acting on it, which we allow to depend on the position and velocity of the particle, and on the time t. When written out in components, (1.1) becomes a system of three simultaneous second-order differential equations,

$$\begin{aligned}\ddot{x} &= F_1(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)\\ \ddot{y} &= F_2(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)\\ \ddot{z} &= F_3(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)\end{aligned}$$

which determine the three functions $x(t)$, $y(t)$, $z(t)$ in terms of the initial position and velocity.

There is considerable freedom in the choice of the inertial frame. We could pick a different origin or we could choose new directions for the axes. We could also replace R by a second frame $\tilde{R}$ moving relative to R without rotation and at constant velocity. Then the particle would have the same acceleration relative to $\tilde{R}$ and (1.1) would still hold.

If we ignore the effects of the Earth's rotation and acceleration, then a set of axes fixed on the Earth's surface is an inertial frame. But, again ignoring the effects of rotation and acceleration, Newton's laws are equally valid on the Moon, which is moving relative to the Earth at about one kilometre per second, or on the sun (about 30 km s^{-1}), or in the Andromeda galaxy (about 270 km s^{-1}). Only an extreme geochauvinist would insist on giving special status to terrestrial frames. As far as mechanical problems are concerned, all non-accelerating, non-rotating frames must be treated equally. To develop this idea in detail, we need to understand the relationship between coordinates and vector components measured in different frames of reference.

Exercise 1.1

Count the number of degrees of freedom in each of the following systems.

(a) A small bead sliding on a wire.

(b) A lamina moving in its own plane.

(c) A double pendulum confined to a vertical plane. This consists of a point mass A suspended from a fixed point by a thin rod; and a second point mass B suspended from A by a second thin rod. The rods are hinged at A.

(d) A double pendulum which is not confined to a vertical plane.

1.2 Frames of Reference

A general frame of reference consists of an origin and a set of right-handed Cartesian coordinate axes. These should not be regarded as fixed: the origin can be a moving point and the axes can be rotating.

Such a frame is represented by a pair $R = (O, \mathcal{T})$, where O is the origin and $\mathcal{T} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the triad of unit vectors along the coordinate axes. The $\boldsymbol{e}_i$s satisfy three conditions. First,

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1$$

at all times because they are unit vectors. Second,

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0$$

because the Cartesian axes are orthogonal, and, third,

$$\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \wedge \boldsymbol{e}_3) = 1 \tag{1.2}$$

because the axes are right-handed. The first two conditions can be combined in the more compact form

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}, \tag{1.3}$$

for $i, j = 1, 2, 3$. Here δ_{ij} is the *Kronecker delta symbol*, defined by

$$\begin{aligned}&\delta_{11} = \delta_{22} = \delta_{33} = 1\\&\delta_{12} = \delta_{21} = \delta_{23} = \delta_{32} = \delta_{31} = \delta_{13} = 0.\end{aligned}$$

Three vectors satisfying (1.2) and (1.3) at all times are said to make up a right-handed *orthonormal triad*, which we shorten to ‘orthonormal triad’, taking ‘right-handed’ as understood.

Any vector $\boldsymbol{x}$ can be expressed as a linear combination of the triad vectors in the form

$$\boldsymbol{x} = x_1\boldsymbol{e}_i + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3, \tag{1.4}$$

where the x_is are functions of time, called the $\mathcal{T}$*-components* of $\boldsymbol{x}$. By taking the dot product with each triad vector in turn,

$$x_1 = \boldsymbol{x} \cdot \boldsymbol{e}_1, \quad x_2 = \boldsymbol{x} \cdot \boldsymbol{e}_2, \quad x_3 = \boldsymbol{x} \cdot \boldsymbol{e}_3.$$

A point P can be specified by its O*-position vector*, which is the vector from O to P. The components of this vector are the Cartesian coordinates of P in the frame.

In thinking about the relationship between vector components and coordinates in different frames of reference, one should keep in mind that a vector is not localized at a point. It is simply a quantity with magnitude and direction. If the distance from A to B is the same as the distance from $\tilde{A}$ to $\tilde{B}$, and if AB is parallel to $\tilde{A}\tilde{B}$, then the vector from A to B is the same as the vector from $\tilde{A}$ to $\tilde{B}$.

1.3 Transition Matrices

Suppose that $\mathcal{T} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ and $\tilde{\mathcal{T}} = (\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3)$ are two orthonormal triads, which may be rotating relative to each other. Then, for $i, j = 1, 2, 3$,

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} = \tilde{\boldsymbol{e}}_i \cdot \tilde{\boldsymbol{e}}_j$$

at all times. Put $H_{ij} = \boldsymbol{e}_i \cdot \tilde{\boldsymbol{e}}_j$ and put

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}. \tag{1.5}$$

Definition 1.7

The matrix H is the *transition matrix* from $\tilde{\mathcal{T}}$ to $\mathcal{T}$.

Note the transition matrix from $\mathcal{T}$ to $\tilde{\mathcal{T}}$ is the transposed matrix H^{t}. The H_{ij}s are nine functions of time, labelled by i and j. They determine the relative orientation of the two triads. With i fixed, H_{i1}, H_{i2}, H_{i3} are the $\tilde{\mathcal{T}}$-components of $\boldsymbol{e}_i$. With j fixed, H_{1j}, H_{2j}, H_{3j} are the $\mathcal{T}$-components of $\tilde{\boldsymbol{e}}_j$. Thus

$$\begin{aligned} \boldsymbol{e}_i &= H_{i1}\tilde{\boldsymbol{e}}_1 + H_{i2}\tilde{\boldsymbol{e}}_2 + H_{i3}\tilde{\boldsymbol{e}}_3 \\ \tilde{\boldsymbol{e}}_i &= H_{1i}\boldsymbol{e}_1 + H_{2i}\boldsymbol{e}_2 + H_{3i}\boldsymbol{e}_3, \end{aligned} \tag{1.6}$$

for $i = 1, 2, 3$.

Example 1.8

Suppose that

$$\begin{aligned} \tilde{\boldsymbol{e}}_1 &= \cos\theta\, \boldsymbol{e}_1 - \sin\theta\, \boldsymbol{e}_2 \\ \tilde{\boldsymbol{e}}_2 &= \sin\theta\, \boldsymbol{e}_1 + \cos\theta\, \boldsymbol{e}_2 \\ \tilde{\boldsymbol{e}}_3 &= \boldsymbol{e}_3 \end{aligned}$$

Then

$$H = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \boldsymbol{e}_1 &= \cos\theta\, \tilde{\boldsymbol{e}}_1 + \sin\theta\, \tilde{\boldsymbol{e}}_2 \\ \boldsymbol{e}_2 &= -\sin\theta\, \tilde{\boldsymbol{e}}_1 + \cos\theta\, \tilde{\boldsymbol{e}}_2 \\ \boldsymbol{e}_3 &= \tilde{\boldsymbol{e}}_3. \end{aligned}$$

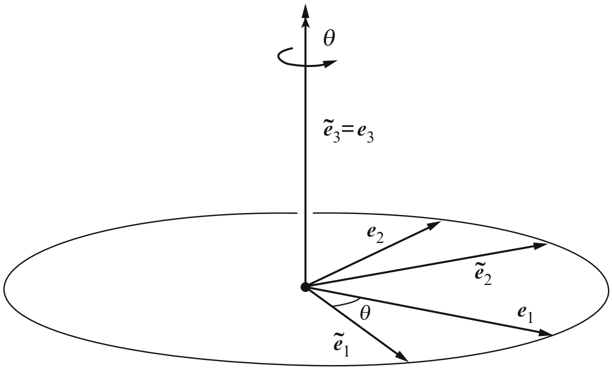

Figure 1.3

The triad $\mathcal{T}$ is obtained from $\tilde{\mathcal{T}}$ by rotation through θ about an axis parallel to $\tilde{\boldsymbol{e}}_3$ (Figure 1.3). In the same way, rotations through θ about the axes parallel to $\tilde{\boldsymbol{e}}_1$ and $\tilde{\boldsymbol{e}}_2$ are given respectively by the transition matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \qquad \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{1.7}$$

Now let $\boldsymbol{x}$ be a general vector and put $x_i = \boldsymbol{x}\cdot\boldsymbol{e}_i$, and $\tilde{x}_i = \boldsymbol{x}\cdot\tilde{\boldsymbol{e}}_i$. The x_is are the $\mathcal{T}$-components of $\boldsymbol{x}$ and the $\tilde{x}_i$s are the $\tilde{\mathcal{T}}$-components of $\boldsymbol{x}$. By substituting from (1.6), we obtain

$$x_i = \boldsymbol{x}\cdot\boldsymbol{e}_i = \sum_{j=1}^{3} H_{ij}\boldsymbol{x}\cdot\tilde{\boldsymbol{e}}_j = \sum_{j=1}^{3} H_{ij}\tilde{x}_j$$

and

$$\tilde{x}_i = \boldsymbol{x}\cdot\tilde{\boldsymbol{e}}_i = \sum_{j=1}^{3} H_{ji}\boldsymbol{x}\cdot\boldsymbol{e}_j = \sum_{j=1}^{3} H_{ji}x_j$$

or, in matrix notation,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = H\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = H^{\mathrm{t}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{1.8}$$

where the ‘t’ denotes transposition. Because this holds for any $\boldsymbol{x}$, H^{t} must also be the inverse of H. Hence

$$H^{\mathrm{t}}H = I = HH^{\mathrm{t}},$$

where I is the 3×3 identity matrix. In other words, H is an orthogonal matrix. Exercise (1.7) contains an outline of a demonstration that the right-handedness of the two triads implies that H is also *proper* in the sense that $\det(H) = 1$.

EXERCISES

1.2. Suppose that the matrix H in (1.5) is given by

$$H = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ -2 & 1 & 1 \\ 0 & -\sqrt{3} & \sqrt{3} \end{pmatrix}$$

Check that $H^{\mathrm{t}}H = HH^{\mathrm{t}} = I$. Write down the components of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ in $\tilde{\mathcal{T}}$, and the components of $\tilde{\boldsymbol{e}}_1$, $\tilde{\boldsymbol{e}}_2$, and $\tilde{\boldsymbol{e}}_3$ in $\mathcal{T}$.

1.3. What are the transition matrices for rotations through $\pm 2\pi/3$ about an axis aligned with the vector with $\mathcal{T}$-components $(1, 1, 1)$?

1.4. Show that if H is a proper orthogonal matrix such that $H_{33} = 1$, then there is a unique angle $\alpha \in [0, 2\pi)$ such that

$$H = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that if $H_{33} = -1$, then there is a unique angle $\alpha \in [0, 2\pi)$ such that

$$H = \begin{pmatrix} -\cos\alpha & -\sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Sketch a diagram of two orthonormal triads with this transition matrix, showing the angle α.

1.5. Show that if H is an orthogonal matrix, then $H^{\mathrm{t}}(H - I) = (I - H)^{\mathrm{t}}$. Deduce that if H is also proper, then $\det(I - H) = 0$. Hence show that if $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are two (right-handed) orthonormal triads, then there exists a nonzero vector that has the same components in both triads.

1.6. Show that if $\tilde{\boldsymbol{e}}_i = \sum_j H_{ji}\boldsymbol{e}_j$ where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is an orthonormal triad, then

$$\tilde{\boldsymbol{e}}_1 \cdot (\tilde{\boldsymbol{e}}_2 \wedge \tilde{\boldsymbol{e}}_3) = \det(H)\, \boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \wedge \boldsymbol{e}_3).$$

Deduce that if $\tilde{\mathcal{T}} = (\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3)$ and $\mathcal{T} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ are right-handed orthonormal triads, then the transition matrix from $\tilde{\mathcal{T}}$ to $\mathcal{T}$ is a *proper* orthogonal matrix.

1.4 Euler Angles

Let $\mathcal{T}$ a right-handed orthonormal triad and let H denote the transition matrix from a second right-handed orthonormal triad $\tilde{\mathcal{T}}$ to $\mathcal{T}$. The nine entries in H determine the relative orientation of the two triads, but we cannot specify them independently of each other because H must satisfy the orthogonality condition

$$HH^{\mathrm{t}} = I.$$

There are six independent equations here, for example, three for the diagonal entries and three for the entries above the diagonal, so we should be able to express H in terms of three independent parameters. The following proposition shows that this is indeed the case.

Proposition 1.9

Let H be a 3×3 proper orthogonal matrix. Then there exist angles $\theta \in [0, \pi]$, $\varphi, \psi \in [0, 2\pi)$ such that

$$H = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Moreover ψ, θ, and φ are uniquely determined by H provided that $|H_{33}| \neq 1$.

Proof

First we deal with uniqueness. If ψ, θ, and φ exist, then

$$\begin{aligned} H_{33} &= \cos\theta, & & \\ H_{31} &= \sin\theta\cos\varphi, & H_{32} &= \sin\theta\sin\varphi, \\ H_{13} &= -\sin\theta\cos\psi, & H_{23} &= \sin\theta\sin\psi. \end{aligned} \tag{1.9}$$

The first equation fixes the value of θ uniquely in the interval $[0, \pi]$. The second pair then determine φ uniquely in $[0, 2\pi)$, provided that $\sin\theta \neq O$, that is, provided that $|H_{33}| \neq 1$. Finally, under the same condition, the last pair determine ψ uniquely in $[0, 2\pi)$.

To establish existence, we consider first the case $|H_{33}| \neq 1$. Choose φ and ψ so that

$$H_{31}\sin\varphi - H_{32}\cos\varphi = 0, \qquad H_{13}\sin\psi + H_{23}\cos\psi = 0.$$

These determine φ and ψ up to the addition of integral multiples of π. Consider the matrix K defined by

$$K = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} H \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.10}$$